

FUNCTIONAL DETERMINANTS ON MÖBIUS CORNERS

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1. Historical remarks.

My title and subject matter are entirely appropriate for a talk delivered in Leipzig where Möbius spent most of his distinguished academic career. Furthermore, the fundamental quantum field theory paper by Heisenberg and Euler originated here and the classic book by Pockels on the Helmholtz equation was published in 1891 by Teubner's in their famous Handbuch series.

2. Brief motivation.

A functional determinant is the determinant of an operator, thus extending the standard notion from the finite to the infinite dimensional situation. The reason for its physical importance is that the one-loop effective action, or the effective action in a background field, is determined, in the simplest case, by the determinant of the propagating (differential) operator.

In such, or equivalent, language this result dates to Feynman, Neumann and Schwinger in the 40's and 50's although the idea, and evaluation, of an effective action is much older and can be traced back to the beginnings of quantum field theory (QED) being associated with Uehling, Serber, Heisenberg & Euler and Weisskopf. Nowadays, the most popular way of doing things in field, instanton and string theory is via path-integration.

If the background field is a gravitational one, the determinant is a function of the metric and therein lies its mathematical importance. Usually the operator is a geometric one such as the Laplacian, and the determinant is a *geometric invariant*. In fact the alternating sum (with coefficients) of form determinants on a manifold is a *topological invariant* – the *analytic torsion*.

Functional determinants have been employed in connection with the uniformisation theorem of Riemann surfaces and with the isospectral problem. The functional determinant depends on, and is determined by, the eigenvalues of the operator. It is thus a *spectral invariant*.

3. The zeta function.

A convenient but by no means the only way of organising the eigenvalues is via the Minakshisundaram ζ -function,

$$\zeta(s) = \sum \frac{1}{\lambda^s}$$

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so that formally (*à la* Euler),

$$\text{Det } D = \prod \lambda = \exp \left(\sum \ln \lambda \right) = e^{-\zeta'(0)}$$

and I will hereafter consider the problem of finding the functional determinant as synonymous with finding $\zeta'(0)$. My emphasis is already turning to the mathematical side.

The central quantity in this approach is thus the ζ -function. The sum definition converges only for s in a certain region, typically $s > d/2$ where d is the dimension of the manifold \mathcal{M} , if D is the Laplacian for example. This makes a continuation to $s = 0$ (and may be to other values) necessary.

One more standard thing. The heat-kernel, K (the proptime kernel of Fock, Nambu, Feynman and Schwinger), and the ζ -function are related by a Mellin transform and contain the same information, modulo zero modes. Some formulae will follow.

The coefficients in the short-time expansion of K are important quantities determining the divergences of the field theory and its scaling behaviour. If D is a geometric operator, they are geometric invariants. Some explicit general forms exist for the early coefficients and specific algorithms for determining all the coefficients can be found in the contribution by Schimming to this conference.

The analytic structure, *i.e.* the poles, and the values of the ζ -function at specific s , is related to these coefficients whose values in any particular case can then be found from the ζ -function. They can be checked against the general forms, or, more usefully, can be used in the determination of any unknown coefficients.

The calculations I shall later report on are to be taken in this spirit of ‘Special Case Evaluation’.

We will hear more things about the ζ -function in the following talk by Elizalde.

4. Summary of methods.

A number of possible approaches are available for finding the ζ -function.

First, if the eigenvalues are known explicitly, *e.g.* $\lambda = n$, one might look at $\sum (1/\lambda^s)$ and seek a continuation, often by reducing the sum, in a more or less direct way, to known (*i.e.* named) ζ -functions, (in the above case to the Riemann ζ -function, $\zeta_R(s)$) for which the continuation has already been done for us. This would be the ideal situation, if all we want is the answer. Alternatively, there might be special function properties, contour representations and summation formulae that can be used, requiring a certain amount of ingenuity to effect the continuation.

Second, if the eigenvalues are known only *implicitly*, the above approach is not possible and one might not be able to find $\zeta(s)$ for all s . However, sufficient information might still be available for our purposes. This is the situation I want to consider in a specific case later.

I will introduce such a method via the modified ζ -function

$$\zeta(s, m^2) = \sum \frac{1}{(\lambda + m^2)^s}$$

which bears the same relation to $\zeta(s)$ that the Hurwitz ζ -function does to the Riemann one and

$$\begin{aligned}\zeta(s, m^2) &= \frac{1}{\Gamma(s)} \int_0^\infty \tau^{s-1} e^{-m^2 \tau} K(\tau) d\tau \\ &= \int_{c-i\infty}^{c+i\infty} ds' (m^2)^{s-s'} \frac{\Gamma(s') \Gamma(s-s')}{\Gamma(s)} \zeta(s', 0),\end{aligned}$$

$d/2 < c < (d+1)/2$. (Dikii.)

$\zeta(s, m^2)$ can be evaluated at an s , say q , where the sum converges, the variable m^2 then providing the access to the required information. (In this sense the construction of $\zeta(q, m^2)$ is an alternative regularisation process to that of continuing $\zeta(s)$. Dikii, Gelfand).

In physical terms, m^2 is a mass and, as such, was separated by Feynman in his proper-time approach. In the heat-kernel, the mass produces convergence at the upper, infinite limit of the Mellin transform allowing one to substitute the short-time expansion of K . This yields an alternative method of deducing the analytic properties of the ζ -function and results in an asymptotic expansion of $\zeta(s, m^2)$, valid for large m^2 , whose coefficients are obviously the heat-kernel expansion coefficients, and provides a method of finding these coefficients (Dikii, Moss). I am not concerned with this aspect here, being more interested in the functional determinant and will present a general method of obtaining this that also involves the asymptotic behaviour as $m \rightarrow \infty$.

5. Weierstrass regularisation.

The first step is to regularise the sum definition of $\zeta(s, m^2)$ in the following way

$$\zeta^*(s, m^2) = \sum^* \frac{1}{(\lambda + m^2)^s} \equiv \sum \left(\frac{1}{(\lambda + m^2)^s} - \frac{1}{\lambda^s} - \sum_{k=1}^M \binom{-s}{k} \frac{m^{2k}}{\lambda^{k+s}} \right)$$

where sufficient terms in the Taylor series, for any particular value of s , have been removed to ensure convergence. I refer to this as *Weierstrass regularisation*, the reason being that if the expression is differentiated with respect to s , and s set to zero I find

$$\zeta^{*'}(0, m^2) = - \sum \left(\ln(1 + m^2/\lambda) + P(m^2/\lambda) \right) = - \sum^* \ln(1 + m^2/\lambda)$$

where the polynomial P is

$$P(x) = x + \frac{x^2}{2} + \dots + \frac{x^{2[d/2]}}{[d/2]}$$

and we recognise

$$e^{-\zeta^{*'}(0, m^2)} = \prod \left(1 + \frac{m^2}{\lambda} \right) e^{P(m^2/\lambda)}$$

as a *Weierstrass canonical product*. In quantum field theory, a modified determinant occurs already in the work of Schwinger.

Let me now perform the summation over λ in $\zeta^*(s, m^2)$ to give the continuation

$$\zeta^*(s, m^2) = \zeta(s, m^2) - \zeta(s, 0) - \sum_{k=1}^M \binom{-s}{k} m^{2k} \zeta(s+k, 0).$$

In particular

$$\zeta^{*'}(0, m^2) = \zeta'(0, m^2) - \zeta'(0, 0) - \frac{\partial}{\partial s} \sum_{k=1}^{[d/2]} \binom{-s}{k} m^{2k} \zeta(s+k, 0) \Big|_{s=0}.$$

Combining these two expressions for $\zeta^{*'}(0, m^2)$, and turning the equation around gives the quantity I am seeking,

$$\zeta'(0, 0) = \zeta'(0, m^2) + \sum^* \ln(1 + m^2/\lambda) - \frac{\partial}{\partial s} \sum_{k=1}^{[d/2]} \binom{-s}{k} m^{2k} \zeta(s+k, 0) \Big|_{s=0}.$$

(*cf* Voros, Quine, Heydari & Song, Jorgenson & Lang.) The last term can be carried further, but I don't need to here.

The essential, rather trivial, point now is that the right-hand side of this equation must be mass-independent and so can be calculated at any convenient value of m , in particular at $m = \infty$. It is not necessary to check that all m -dependence does disappear. One just needs to keep the constant, mass-independent parts of the various asymptotic limits. This is the method.

The asymptotic expansion of $\zeta'(0, m^2)$ referred to earlier shows that

$$\zeta'(0, m^2) \sim 0 + O(\ln m^2)$$

then I arrive at the final, working formula

$$\zeta'(0) \approx \lim_{m \rightarrow \infty} \sum^* \ln(1 + m^2/\lambda)$$

or

$$\lim_{m \rightarrow \infty} \sum^* \ln(1 + m^2/\lambda) = \zeta'(0) + O(\ln m).$$

This is a perfectly general equation. I want to illustrate the method by applying it to the situation where the eigenvalues are known only implicitly. Instead of giving a general treatment, I proceed now to the specific calculation I wish to report on at this meeting.

I should say that related but different, methods have been developed by Barvinsky, Kamenshchik, Karmazin and Mishakov and by Bordag, Geyer, Kirsten and Elizalde, a group based mostly here at Leipzig.

6. The geometry and the calculation.

Apart from any physical motivation, the plan is to look for geometrical domains on which there is sufficient knowledge of the eigenvalue problem to enable the functional

determinant to be evaluated in ‘closed form’. The operator is the Laplacian on scalars. Other fields can be treated of course but I just want to illustrate the technique. Pockels’ book contains valuable information on this eigenvalue problem. (Incidentally, this book contains probably the first discussion of the eigenvalue counting function.)

The basic geometry I have in mind is that of the *Euclidean* $(d + 1)$ -ball. Although I have been at pains to say that I have no immediate physical motivation in mind, I should point out that a considerable amount of work has been performed on such, and related, geometries in the context of quantum cosmology by *e.g.* D’Eath, Esposito, Schleich, Moss, Pollifrone, Vassilevich and the Russian group, so my calculation may not be entirely sterile.

The functional determinants (and heat-kernel coefficients) on the full ball have been determined by the Leipzig group in the scalar case, and by Kirsten and Cognola for other fields using methods that parallel, but are in detail different to, my own. Of course, I like my technique better.

Actually I will not be dealing with the full ball here, but rather with a *bounded, generalised cone* of a special sort. This is, loosely speaking, the solid angle subtended at the origin of the ball by a finite domain on its surface. It is not possible to treat arbitrary shapes, I think, and I shall be restricted to domains with special symmetry associations which I will describe in a moment.

The reason for the calculation is purely one of aesthetics and not governed by any physical motivation. At the formal level, I wish to advertise the usefulness of the Barnes ζ - and multiple Γ -functions.

The bounded, generalized cone is defined as the space $\mathbb{R}^+ \times \mathcal{N}$ with the hyperspherical metric

$$ds^2 = dr^2 + r^2 d\Sigma^2$$

where $d\Sigma^2$ is the metric on the manifold \mathcal{N} , and r runs from 0 to 1. For us, $\mathcal{N} \sim S^d$ locally and $d\Sigma^2$ is the metric on the d -sphere. (*cf* Cheeger.)

To set up the formalism, I outline some very standard textbook material. The Laplacian can be written

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_S$$

where Δ_S here is the Laplacian on the sphere.

The eigenmodes of Δ , with eigenvalue $-\alpha^2$, are of the form

$$\frac{J_\nu(\alpha r)}{r^{(d-1)/2}} Y(\Omega)$$

where the harmonics on \mathcal{N} satisfy

$$\Delta_S Y(\Omega) = -\bar{\lambda}^2 Y(\Omega)$$

and

$$\nu^2 = \bar{\lambda}^2 + (d - 1)^2/4.$$

At this point I restrict the region \mathcal{N} to be a fundamental domain of the complete symmetry group, Γ , of a $d + 1$ -dimensional polytope acting on its circumscribing sphere

S^d . I have used such regions several times before and, following Terras, have referred to them as *Möbius corners*, $\mathcal{C}(\mathbf{d})$. In the present case, when the range of r is limited, the corner will be a bounded one. An interesting aspect of these spaces is that their boundaries are only piecewise smooth.

In three dimensions, the infinite corner can be realised as three planar mirrors intersecting at one point to form a *trihedral kaleidoscope* (Möbius' term). To qualify as a genuine Möbius corner, the dihedral angles must be such as to allow an integral number of similar corners to fill out the entire 4π solid angle when arranged around a common point of intersection.

Incidentally, if the domains are identified to form an orbifold, their edges (running through the origin) form three angular defects (cosmic strings). The Casimir effect has been calculated in the infinite corner. (Dowker and Chang.)

The eigenfunctions and eigenvalues on \mathcal{N} are determined by group and invariant theory. The usual spherical harmonics are filtered (symmetry adapted) by the condition of being invariant under Γ . The important fact for my calculation is that the eigenvalues $\bar{\lambda}$ are given by

$$\bar{\lambda}_{\mathbf{n}}^2 = (\mathbf{n} \cdot \mathbf{d} + a)^2 - (d-1)^2/4$$

where \mathbf{d} is a d -dimensional vector of the integer degrees associated with the tiling group, Γ , and where the integers n_i range from zero to infinity. The degeneracy of any particular eigenvalue depends on the number of coincidences as \mathbf{n} varies. The parameter a determines the type of boundary conditions on the *sides* of the cone and, in what is effectively an image-based approach, I can, unfortunately, treat only Dirichlet and Neumann conditions with a equalling $\sum d_i - (d-1)/2$ and $(d-1)/2$ respectively.

You now see the nice circumstance that the order of the Bessel function is still an integer or a half odd-integer, depending on the oddness or evenness of the dimension d ,

$$\nu_{\mathbf{n}} = \mathbf{n} \cdot \mathbf{d} + a.$$

The boundary condition on the spherical end of the cone at $r = 1$ has still to be fixed. Just as for the full ball, one can impose Dirichlet, Neumann or Robin conditions. For Dirichlet the implicit eigenvalue condition is

$$J_{\mathbf{n} \cdot \mathbf{d} + a}(\alpha) = 0,$$

while Robin boundary conditions yield,

$$\beta J_{\mathbf{n} \cdot \mathbf{d} + a}(\alpha) + \alpha J'_{\mathbf{n} \cdot \mathbf{d} + a}(\alpha) = 0,$$

where β is a constant. (Actually β could be a function on \mathcal{N} but I won't discuss this here.)

In order to extract the eigenvalue properties from these equations, I follow previous workers and use Euler's representation of the Bessel function in terms of its zeros, to write, for example,

$$\sum_p \ln(2^p p! m^{-p} I_p(m)) = \sum_{p, \alpha_p} \ln\left(1 + \frac{m^2}{\alpha_p^2}\right) = \sum_{\lambda} \ln\left(1 + \frac{m^2}{\lambda}\right).$$

Then

$$\zeta'(0) \approx \lim_{m \rightarrow \infty} \sum_p^* \ln(2^p p! m^{-p} I_p(m)),$$

where $p = \mathbf{n} \cdot \mathbf{d} + a$ and the sum over p means a multiple sum over the integers \mathbf{n} .

The further analysis of this expression involves some algebra and will be of interest only for those currently working in this field. However let me give you a flavour of the calculation.

The asymptotic behaviour of the Bessel function is known (Olver) and the $\ln p!$ term can be replaced by an integral representation. Then

$$\begin{aligned} \zeta'(0) \sim \sum_{\mathbf{n}}^* \left[p \ln \frac{2p}{p + \epsilon} + (\epsilon - p) - \frac{1}{2} \ln \frac{\epsilon}{p} + \sum_{l=1}^{\infty} \frac{T_l(t)}{\epsilon^l} \right. \\ \left. + \int_0^{\infty} \left(\frac{1}{2} - \frac{1}{\tau} + \frac{1}{e^{\tau} - 1} \right) \frac{e^{-\tau p}}{\tau} d\tau \right], \end{aligned} \quad (1)$$

where the $T_l(t)$ are Olver's polynomials in $t = p/\epsilon$ and $\epsilon^2 \equiv p^2 + m^2$.

I will indicate how to deal with the first three terms. One simply applies the Weierstrass regularisation again. Thus

$$\sum_{\mathbf{n}}^* \frac{p^N}{\epsilon^{2s}} = \sum_{\mathbf{n}} p^N \left[\frac{1}{\epsilon^{2s}} - \frac{1}{p^{2s}} - \sum_{h=1}^M \binom{-s}{h} \frac{m^{2h}}{p^{2h+2s}} \right] \quad (2)$$

from which, if s is given certain values, possibly after differentiating, the required limits can be found.

I remind you that $p = \mathbf{n} \cdot \mathbf{d} + a$. If the sums are done you can see that an important role is played by the Barnes ζ -function defined by

$$\begin{aligned} \zeta_d(s, a | \mathbf{d}) &= \frac{i\Gamma(1-s)}{2\pi} \int_L d\tau \frac{\exp(-a\tau)(-\tau)^{s-1}}{\prod_{i=1}^d (1 - \exp(-d_i\tau))} \\ &= \sum_{\mathbf{n}=0}^{\infty} \frac{1}{(a + \mathbf{n} \cdot \mathbf{d})^s}, \quad \text{Re } s > d. \end{aligned}$$

By general theory, the asymptotic limit of the first term in (2) is

$$\sum_{\mathbf{n}} \frac{1}{((\mathbf{n} \cdot \mathbf{d} + a)^2 + m^2)^s} \sim \sum_{k=0, 1/2, 1, \dots} (m^2)^{d/2-k-s} \frac{\Gamma(s - d/2 + k)}{\Gamma(s)} C_k$$

where the C_k are the coefficients in the short-time expansion of the heat-kernel associated with the Barnes ζ -function, $\zeta_d(2s, a | \mathbf{d})$ *e.g.*

$$(-1)^k k! C_{d/2+k} = \zeta_d(-2k, a | \mathbf{d}),$$

with

$$\zeta_d(-q, a | \mathbf{d}) = \frac{(-1)^d q!}{(d+q)! \prod d_i} B_{d+q}^{(d)}(a | \mathbf{d})$$

in terms of generalized Bernoulli functions.

Actually I do not need these explicit expressions. Applying the Weierstrass regularisation only the second term in (2) contributes and one easily obtains the useful limits

$$\begin{aligned}\sum_{\mathbf{n}} p^N (\epsilon - p) &\sim -\zeta_d(-N-1, a | \mathbf{d}) + O(\ln m), \\ \sum_{\mathbf{n}} p^N \ln\left(\frac{2p}{p+\epsilon}\right) &\sim -\zeta'_d(-N, a | \mathbf{d}) + \ln 2 \zeta_d(-N, a | \mathbf{d}) + O(\ln m), \\ \sum_{\mathbf{n}} p^N \ln\left(\frac{\epsilon}{p}\right) &\sim \zeta'_d(-N, a | \mathbf{d}) + O(\ln m),\end{aligned}$$

enabling the first three terms in equation (1) to be found. I won't inflict the remainder of the calculation on you, although a number of interesting technical points arise. I will just write down the final answer, which is reasonably compact,

$$\begin{aligned}\zeta'_{\mathcal{C}(\mathbf{d})}(0) &= \zeta'_{d+1}(0, a+1 | \mathbf{d}, 1) + \ln 2 \left(\zeta_d(-1, a | \mathbf{d}) + \sum_{l=1}^d T_l(1) N_l(d) \right) \\ &\quad + \sum_{l=1}^d N_l(d) \int_0^1 t^{l-1} T_l''(t) dt + \frac{1}{2} \sum_{l=1}^d T_l(1) N_l(d) \sum_{k=1}^{(l-1)/2} \frac{1}{k}.\end{aligned}$$

The $N_l(d)$ are the residues of the Barnes ζ -function and, like the $T_l(1)$, are given by Bernoulli functions.

Everything is quite explicit, apart from the derivative of the Barnes ζ -function which, in any particular case, could be reduced to a number of Hurwitz ζ -functions. I have thus achieved my aim of finding a closed form. More explicit expressions exist for the full sphere case and agree with those of the Leipzig group when evaluated at specific dimensions.

7. The Robin case.

A little more involved, and therefore more interesting, is the case of Robin conditions on the spherical end of the cone. I am certainly not going to go through the details systematically but I do want to point out the following little bit of formalism. This is mostly self indulgence but some of you might be interested.

A new term in this case is

$$\sum_{\mathbf{n}}^* \ln \left(1 + \frac{\beta}{\mathbf{n} \cdot \mathbf{d} + a} \right)$$

which is the Weierstrass product associated with the eigenvalues $\mathbf{n} \cdot \mathbf{d} + a$ and can be dealt with according to the general result given earlier which, in this case, relates it to the derivative of the Barnes ζ -function, $\zeta'_d(0, a | \mathbf{d})$. Applying this relation, I find

$$\sum_{\mathbf{n}}^* \ln \left(1 + \frac{\beta}{\mathbf{n} \cdot \mathbf{d} + a} \right) = \ln \left(\frac{\Gamma_d(a)}{\Gamma_d(a+\beta)} \right) + \sum_{l=1}^d \frac{\beta^l}{l!} \psi_d^{(l)}(a),$$

in terms of the multiple Γ - and ψ -functions. In fact this generalises Barnes' product for the multiple Γ -function.

The power series

$$\sum_{\mathbf{n}}^* \ln \left(1 + \frac{\beta}{\mathbf{n} \cdot \mathbf{d} + a} \right) = \sum_{l=d}^{\infty} \frac{(-1)^l \beta^{l+1}}{l+1} \zeta_d(l+1, a | \mathbf{d})$$

is also easily derived.

I give the final determinant expression for completeness,

$$\begin{aligned} \zeta_{\mathcal{C}(\mathbf{d})}'(0, \beta) &= \zeta'_{d+1}(0, a | \mathbf{d}, 1) - \ln \left(\frac{\Gamma_d(a)}{\Gamma_d(a + \beta)} \right) - \sum_{l=2, \dots}^d \frac{\beta^l}{l} N_l(d) \sum_{k=1}^{l/2} \frac{1}{2k-1} \\ &+ \frac{1}{2} \sum_{l=1, 3, \dots}^d R_l(\beta, 1) N_l(d) \sum_{k=1}^{(l-1)/2} \frac{1}{k} + \sum_{l=1}^d N_l(d) \int_0^1 t^{l-1} R_l''(\beta, t) dt \\ &+ \ln 2 \left(\zeta_d(-1, a | \mathbf{d}) + \sum_{l=1, 3, \dots}^d R_l(\beta, 1) N_l(d) \right). \end{aligned}$$

It is not much more complicated than the Dirichlet one. The $R(\beta, t)$ are the relevant Olver asymptotic polynomials (Moss).

Using the definition of the multiple Γ -function and Γ -modular form, ρ , I can rewrite part of this expression,

$$\begin{aligned} \zeta'_{d+1}(0, a | \mathbf{d}, 1) - \ln \left(\frac{\Gamma_d(a)}{\Gamma_d(a + \beta)} \right) &= \ln \left(\frac{\Gamma_{d+1}(a)}{\rho_{d+1}(\mathbf{d}, 1)} \frac{\Gamma_d(a + \beta)}{\Gamma_d(a)} \right) \\ &= \ln \left(\frac{\rho_d(\mathbf{d})}{\rho_{d+1}(\mathbf{d}, 1)} \frac{\Gamma_d(a + \beta)}{\Gamma_{d+1}(a + 1)} \right). \end{aligned}$$

Numerically, the difficulty lies in the evaluation of the multiple Γ -functions and the Γ -modular forms. The definitions are,

$$\lim_{\epsilon \rightarrow 0} \zeta_r'(0, \epsilon | \mathbf{d}) = -\ln \epsilon - \ln \rho_r(\mathbf{d}), \quad \zeta_r'(0, a | \mathbf{d}) = \ln \left(\frac{\Gamma_r(a)}{\rho_r(\mathbf{d})} \right).$$

8. Note on the Barnes ζ -function.

The original papers of Barnes contain a lot of information about this function, which can be looked upon as a multidimensional generalisation of the Hurwitz ζ -function. It forms a happy playground for those of us amused by special functions, recursion formulae, curious identities *etc.*

I will not attempt to summarize this material here. All I want to do is to point out that the Barnes ζ -function arises 'naturally' when considering the d -dimensional harmonic oscillator with operator

$$-\Delta_{\text{HO}} = -\nabla^2 + \sum_{i=1}^d \omega_i^2 x_i^2$$

whose eigenvalues are $\lambda = 2\mathbf{n}.\omega + \sum_i \omega_i$. The corresponding ζ -function is

$$\zeta_{\text{HO}}(s) = \sum_{\mathbf{n}=0}^{\infty} \frac{1}{(2\mathbf{n}.\omega + \sum \omega_i)^s} = \zeta_d(s, \sum \omega_i \mid 2\omega).$$

The diagonal heat-kernel can also be written down immediately

$$K_{\text{HO}}(\mathbf{x}, \tau; \mathbf{x}, 0) = \prod_i \frac{\exp(-\omega_i x_i^2 \tanh \omega_i \tau)}{2\pi \sinh 2\omega_i \tau}.$$

Since a constant magnetic field is more or less mathematically equivalent to a multi-dimensional harmonic oscillator, the Barnes ζ -function will occur here also.

9. Conclusion.

There is no conclusion because I am still pursuing the calculation, trying to make sense of some special cases, and I haven't even mentioned Neumann conditions or other fields like spin-1/2 and Maxwell.

The domains I have been engaged with form a discrete series and it would be nice to have some continuously varying parameter so that a graph or two could be drawn. It is possible to discuss a spherical wedge with an arbitrary opening angle (in which case one can have Robin conditions on the sides) and it may be possible to treat a standard spherical ice-cream cone, *i.e.* one whose surface domain, \mathcal{N} , is a *spherical* d -ball or cap. The Möbius cone can also be truncated at an *inner* radius giving a portion of a *shell*.

10. Added note.

I have learnt at this conference that Weierstrass regularisation has been used by Wipf in an interesting discussion of tunnel determinants.

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